ORBITAL DYNAMICS AND ORBITAL DESIGN – LAMBERT'S PROBLEM

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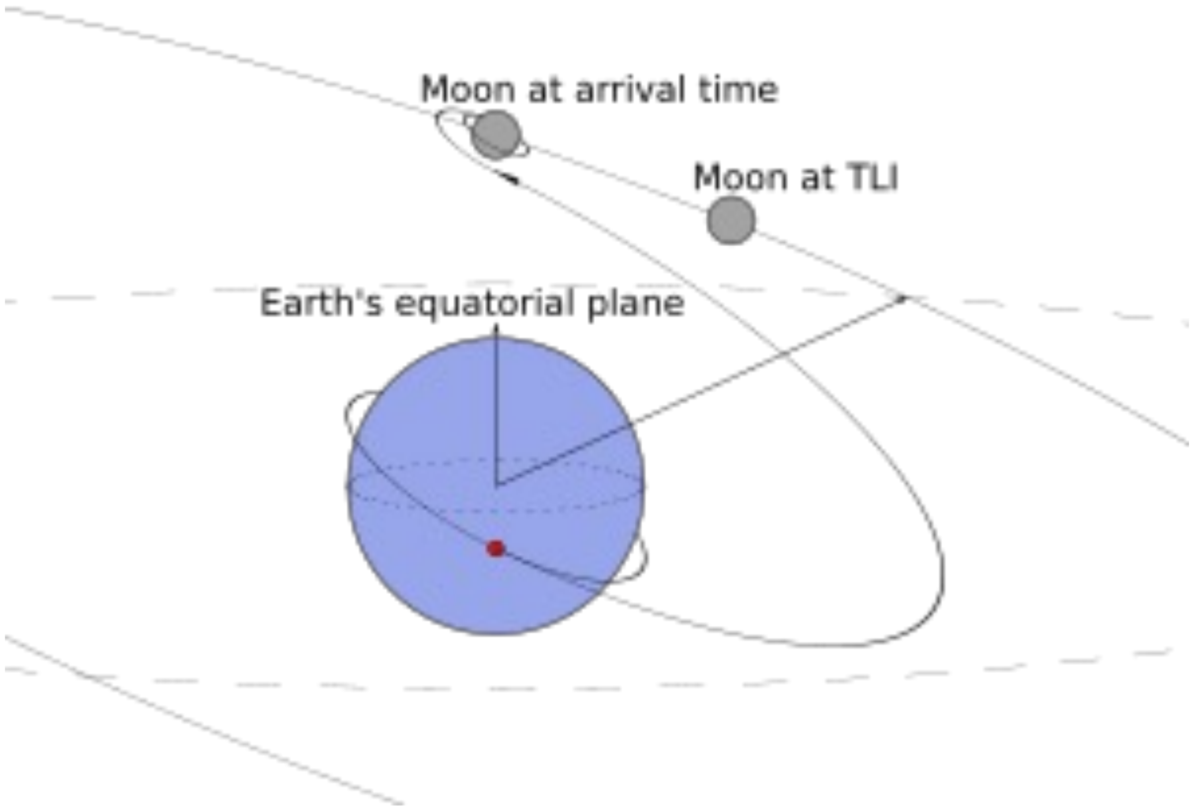
Problem Overview

Initial Conditions:
- Arrive at a lunar orbit on October 28th with a $\pm$ 1day budget
- 8.2 km/s total allotted delta V velocity budget

- *Original Parking Orbit: a = 15 600 km, e = 0.2, i = 75°, RAAN = 0, ω = 0, ν= 0*

- *Final Parking orbit: a = 5 500 km, e = 0.2, i = 110°*



Lamberts Problem - Overview

<u>Lambert's problem asks:</u>

Given two points in space, P1 and P2, under the influence of a gravitational field with parameter μ, and a transfer time ΔT,

Can we determine a unique solution the Keplerian elements of an orbit that connects these points?

If this is possible then:

1. Orbit Determination - what are the Keplerian Elements of an orbit
2. Targeting (two objects reaching the same point in space and time)
3. Rendezvous (two objects reaching the same point in same, time and **velocity**)

Hint.. It is!

(But its not a unique solution)

Given only P1, P2, and ΔT. Only the SMA is the required variable necessary to solve for.

From which we can determine e, i, w, RAAN, TA following this getting the velocity at this point is trivial

Pauca sed matura (Few, but ripe)

Historical Context:

- **Johann Heinrich Lambert** first posed Lambert's Problem in **1761**, formulating the question of determining an orbit given two positions and a time of flight. However, he did not provide a rigorous proof.
- **Joseph-Louis Lagrange** (late 18th century) later analyzed the problem and contributed to its theoretical foundation, but gave no practical solution
- It was **Carl Friedrich Gauss** in **1809** who developed a complete numerical method to solve the problem, correctly predicting the orbital path of "1 Ceres"

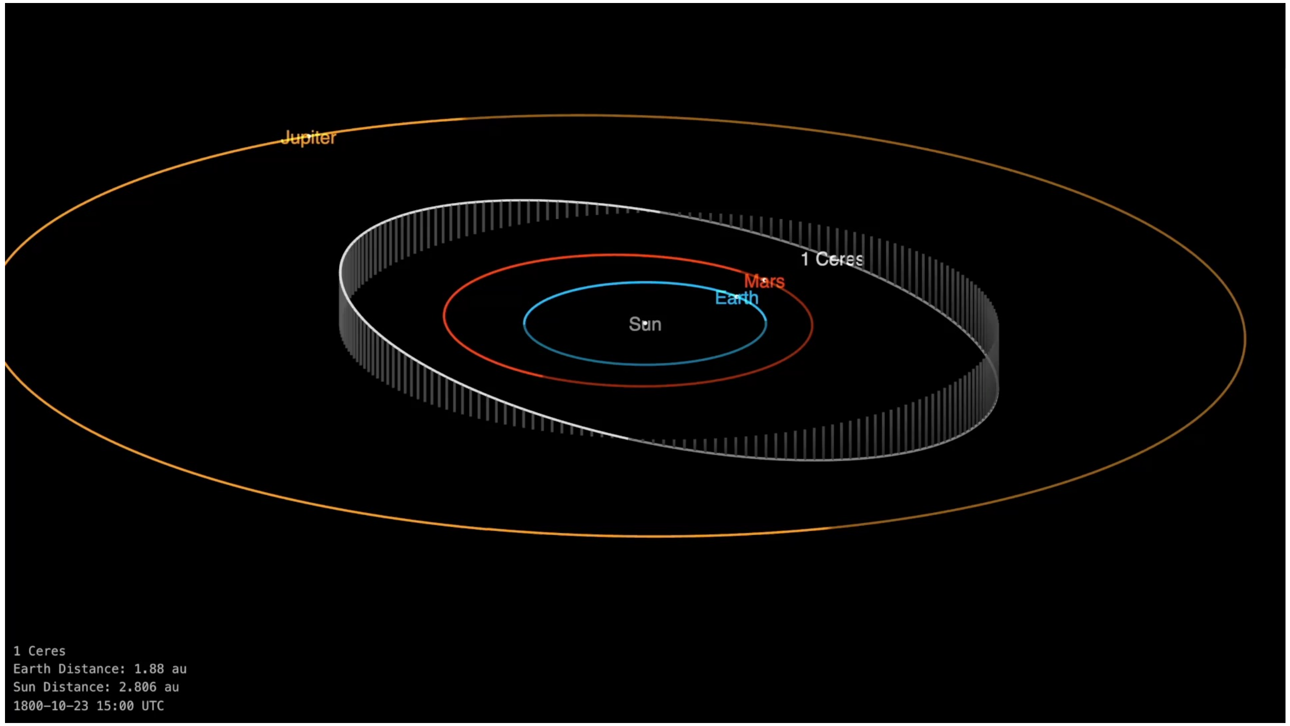
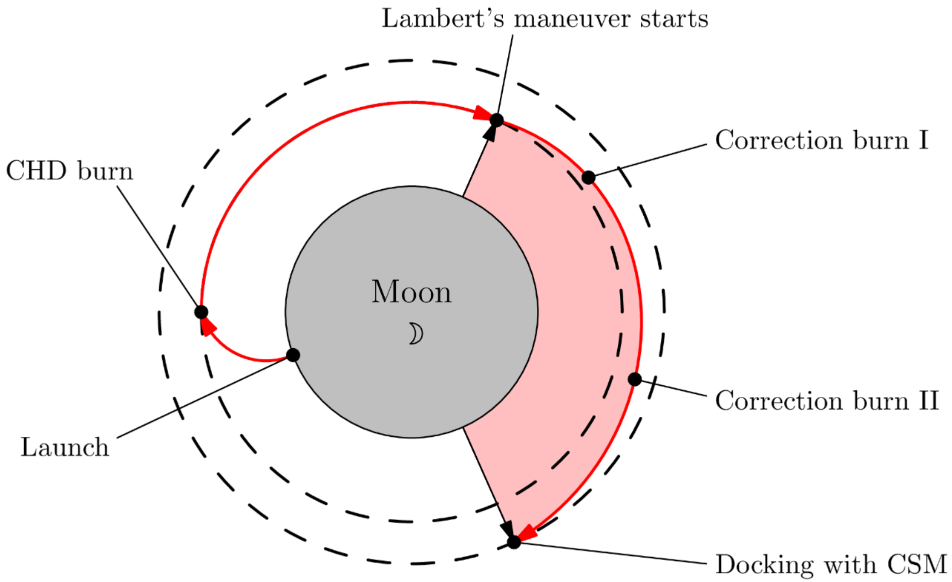


Figure: Position of planets at the time of Gauss 1809

Historical Context:

- The first modern algorithms of Lamberts problem were employed in the 1950 and are currently being developed today. Still an open research question
- Used in the Apolo missions to make the rendezvous with the CSM(Command and Service Module) – reportedly taking 15-30min to perform this calculation




```
# INPUT
# (1) RINIT INITIAL POSITION RADIUS VECTOR
# (2) VINIT INITIAL POSITION VELOCITY VECTOR
# (3) RTARG TARGET POSITION RADIUS VECTOR
# (4) DELLT4 DESIRED TIME OF FLIGHT FROM RINIT TO RTARG
# (5) INTIME TIME OF RINIT
# (6) 0D NUMBER OF ITERATIONS OF LAMBERT/INTEGRVS
# (7) 2D ANGLE TO 180 DEGREES WHEN ROTATION STARTS
# (8) RTX1 -2 FOR EARTH, -10D FOR LUNAR
# (9) RTX2 COORDINATE SYSTEM ORIGIN -- 0 FOR EARTH, 2 FOR LUNAR
# PUSHLOC SET AT 4D
#
# Page 487
# OUTPUT
# (1) RTARG OFFSET TARGET POSITION VECTOR
# (2) VIPRIME MANEUVER VELOCITY REQUIRED
# (3) VTPRIME VELOCITY AT TARGET AFTER MANEUVER
# (4) DELVEET3 DELTA VELOCITY REQUIRED FOR MANEUVER
```

```
# COMPUTE THE DELTA VELOCITY
INITVEL6  VLOAD
          R2VEC
          STORE RTARG1
INITVEL7  VLOAD VSU
          VIPRIME
          VINIT
          STOVL DELVEET3 # DELVEET3 = VIPRIME-VINIT
          (+7)
          VTARGET
          STORE VTPRIME
          SLOAD BHIZ
          RTX2
```

https://github.com/chrislgarry/Apollo-11

Derivation of the Solution to Lamberts Equation

Essentially what we're ultimately looking for is a way to relate time to the true anomaly for a particular orbit.

Luckly this can be done through Kepler's Equation:

$$M = E - e\sin(E)$$

Where:

M is the mean anomaly (should the orbit be circular and not elliptic how far along would we be?)

E is the eccentric anomaly if the orbit were to be an circle where along the path would the true anomaly be

e is the eccentricity

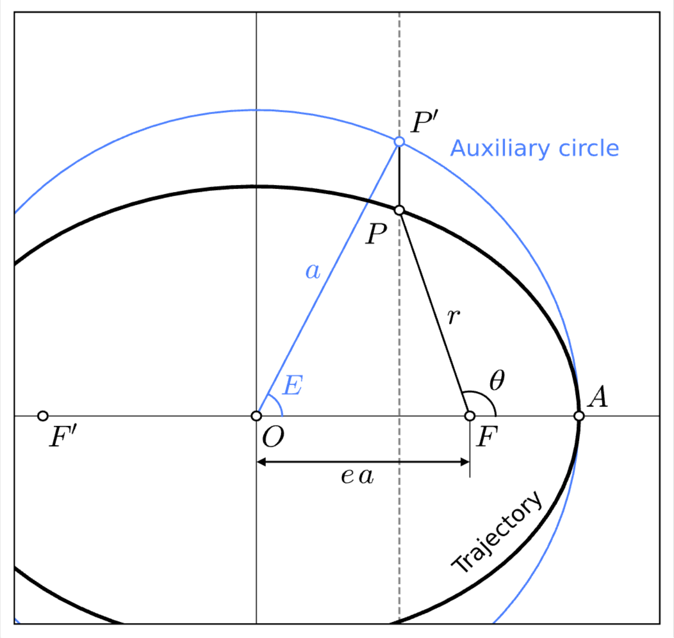


Figure: Mean Eccentricity as it relates to the TA

__Example Use__

$SMA = 10{,}000km,\ t_1 = 0,\ t_2 = 2000s$

$e = 0.3$

(1) Find Angular Frequency

$$n = \sqrt{\frac{\mu}{a^3}}$$

$$n = \sqrt{\frac{3.98 * 10^{14}}{10000 * 10^3}}$$

$$n = 6.308 * 10^{-4} \frac{rad}{sec}$$

(2) Find Mean Anomaly:

$$M = n(t_2 - t_1)$$

$$M = 1.261 \text{ rad}$$

(3) Using the Bisection method we can solve:

$$2.161 = E - 0.3 * \sin(E)$$

$$E = 1.56\ rad$$

(4) Now solve for the True Anomaly θ:

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-e^2} * sinE}{\cos(E-e)}\right)$$

$$\theta = 1.88\ rad$$

Derivation of the Solution to Lamberts Equation

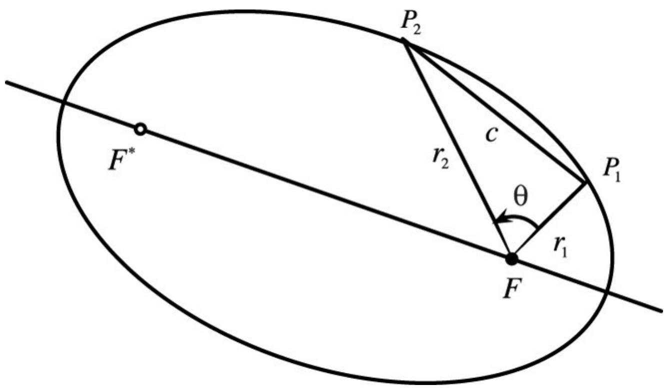
Expanding this to two time measurements:

$$M_1 - M_2 = n[t_1 - t_2 \cancel{- t_p} + \cancel{t_p}] = E_1 - e * \sin(E_1) - (E_2 - e * \sin(E_2))$$

$$(t_2 - t_1) * \sqrt{\frac{\mu}{a^3}} = E_1 - E_2 - e * \sin(E_1) + e * \sin(E_2)$$

This looks useful but its written in terms of Eccentric Anomaly we only have access to the position vectors.
Thus this must be transformed into a form of of *only* $P_1 P_2$ *and* Δt

This can be done geometrically by finding the cord length **C**:



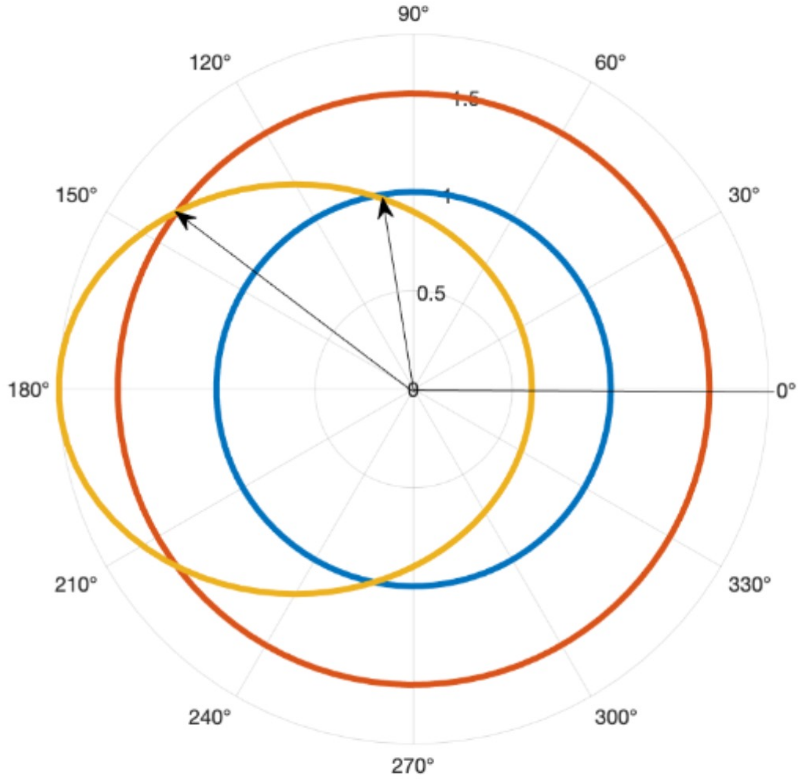


Figure: For given P1 and P2 and TOF, the transfer ellipse is uniquely determined.

Derivation of the Solution to Lamberts Equation

Using $r = a(1 - e * \cos(E)$

and using a substitution of $E_p = \frac{E_1 - E_2}{2}$ $and\ E_M = \frac{E_1 - E_1}{2}$

We find that:

$r_2 + r_1 = 2a\,(1 - e\cos(E_P)\cos(E_M))$

$[*]$ *This is powerful because it directly relates the position vectors to its SMA. But this is unsolvable we need another identity to solve for all the variables here*

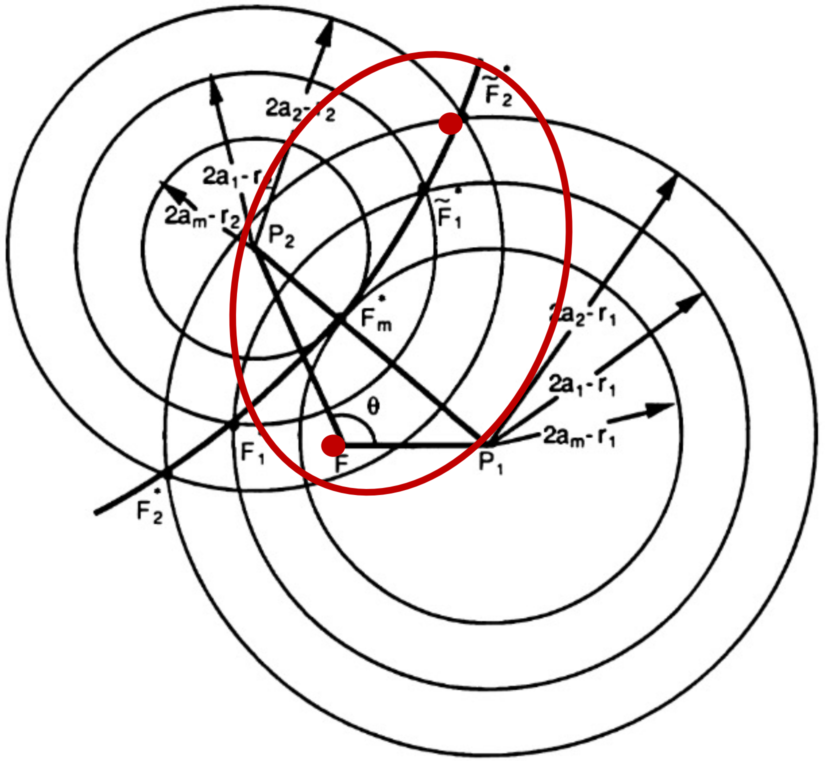


Figure: Geometric Representation of Lambert's Problem and the Role of the Chord Length in Determining the Transfer Orbit

This is done by using the cord distance:

$c = |(\vec{r_1} - \vec{r_2})|$

$$c^2 = 4a^2 * \sin^2(E_M) * (1 - e^2 * \cos^2(E_P))$$

This can be combined to obtain:

$$r_1 + r_2 + c = 2a\,(1 - \cos(\alpha)) = 4a * \sin^2\left(\frac{\alpha}{2}\right)$$

And

$$r_1 + r_2 - c = 2a\,(1 - \cos(\beta)) = 4a * \sin^2\left(\frac{\beta}{2}\right)$$

where:

$\alpha = \cos^{-1}(e * \cos(E_P)) + E_M$

$\beta = \cos^{-1}(e * \cos(E_P)) - E_M$

Derivation of the Solution to Lamberts Equation

Going back now to Kepler's Equation:

$$(t_2 - t_1) * \sqrt{\frac{\mu}{a^3}} = E_1 - E_2 - e * \sin(E_1) + e * \sin(E_2)$$

Incoorprating the geometric properties into leads to:

$$(t_2 - t_1) * \sqrt{\frac{\mu}{a^3}} = [\alpha - \beta - (\sin\alpha - \sin\beta)]$$

where:

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{r_1 + r_2 + c}{4a}}$$

$$\sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{\frac{r_1 + r_2 + c}{2} - c}{2a}}$$

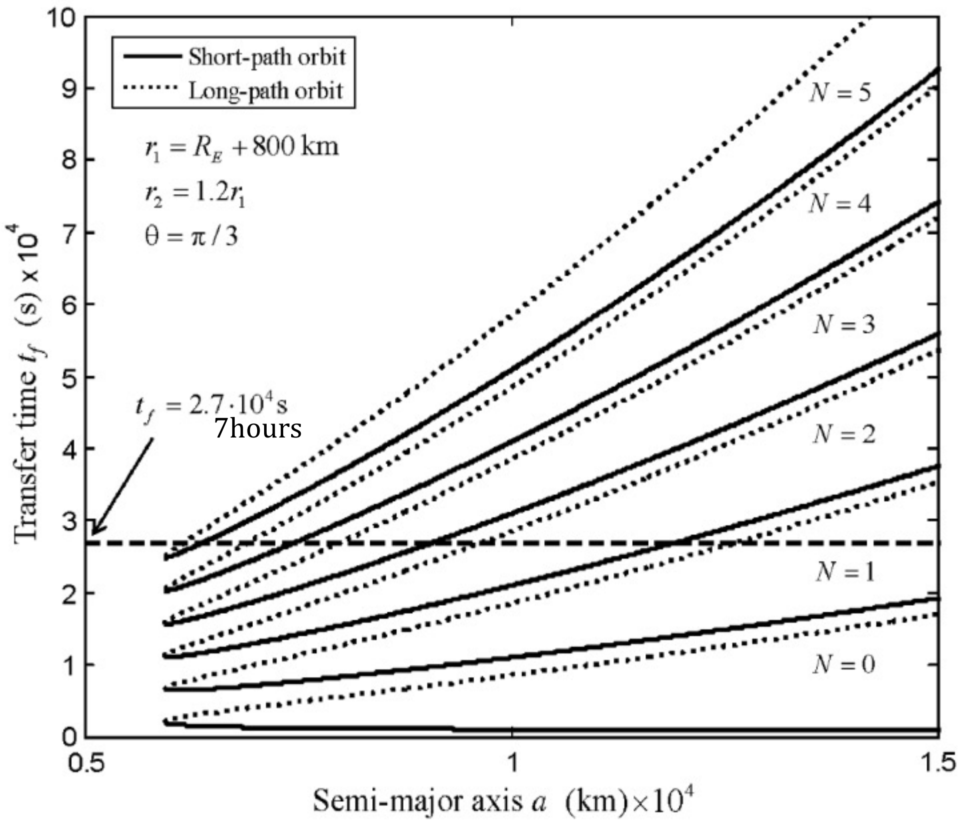
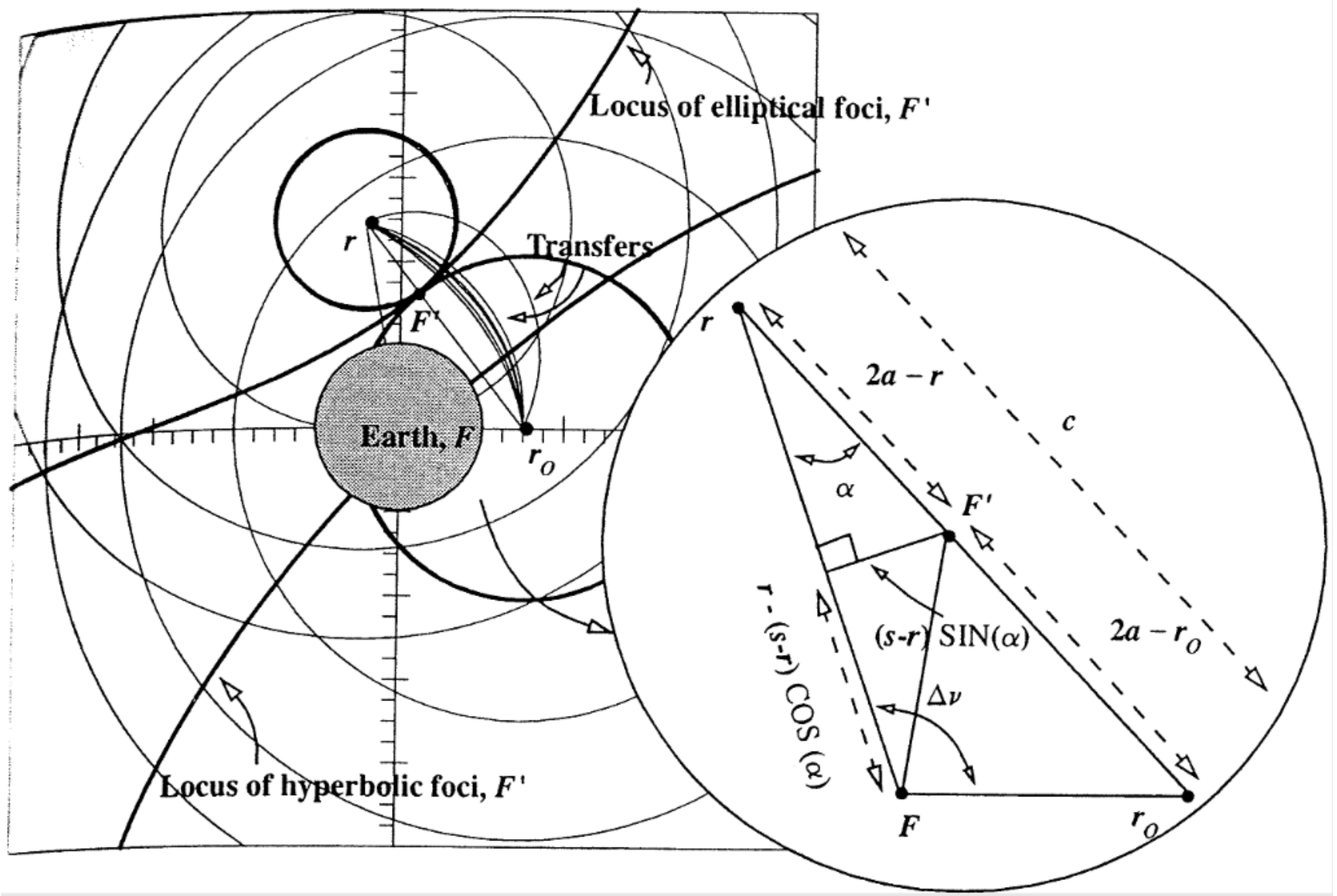


Figure: Solutions to Lambert’s Problem including multiple rotations

Methods for the solution to Lamberts Problem

Minimum Energy Transfer:
Hidden focus for the minimum energy transfer lies on the line between r_1, r_2



Numerical Solutions

Using the Universal solution (Battin 1999)

$$C(z)=\begin{cases}\frac{(1-\cos(\sqrt{z}))}{z}, z>0\\ \frac{\left(\cosh(\sqrt{z})\right)-1}{z}, z<0\\ \frac{1}{2}, z=0\end{cases}\quad S(z)=\begin{cases}\frac{(\sqrt{z}-\sin(\sqrt{z}))}{(\sqrt{z})^3}, z>0\ (Elliptical)\\ \frac{(\sinh\sqrt{-z}-(\sqrt{z}))}{(\sqrt{-z})^3}, z<0\ \ (Hyperbolic)\\ \frac{1}{6}, z=0\ (Parabolic)\end{cases}$$

$$Y(z) = R_1 + R_2 + A \cdot \frac{zS(z)-1}{\sqrt{C(z)}}$$

where: R_1 & R_2 *are the magnatude of the position vectors*
A is a constant related to the transfer angle and geometry

$$F(z,t) = \left(\frac{C(z)}{Y(z)}\right)^{\frac{3}{2}} * S(z) + A\sqrt{Y(z)} - \mu t$$

Goal: Find a z value such that F(z,t) = 0

Solution: Using Newtons Method $z_1 = z_0 - \frac{F(z,t)}{d\,F(z,t)}$

Earth – Mars Launch Windows

By solving Lambert's problem across a range of departure dates and times of flight, we generate a map of possible trajectories.
From which we optimize for parameters such as the required departure velocity (C3), arrival velocity, or total mission Δv.

Only in the shaded areas is where a Ballistic maneuver is possible.

Remember Lambert solvers solve using Keplerian motion.

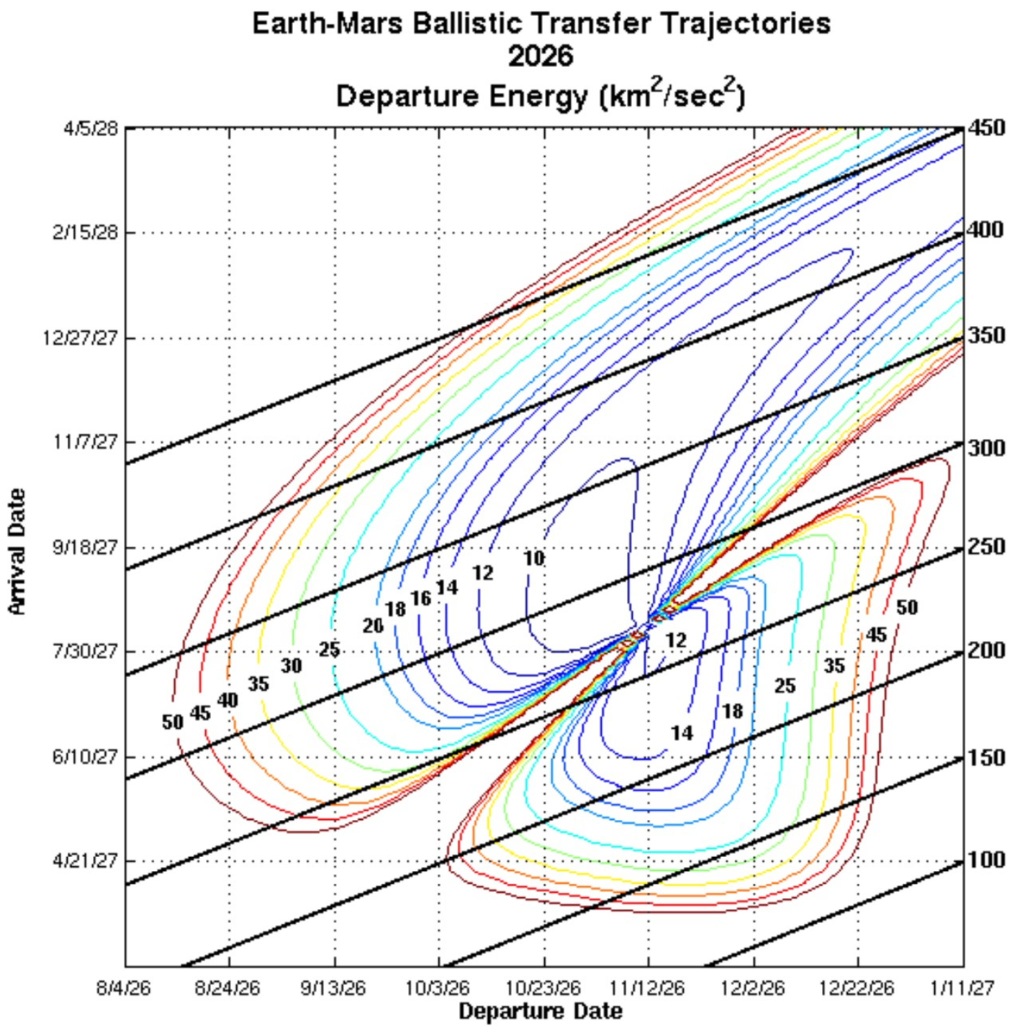


Figure: Porkchop "Interplanetary Mission Design Handbook: Earth-to-Mars Mission Opportunities 2026 to 2045"

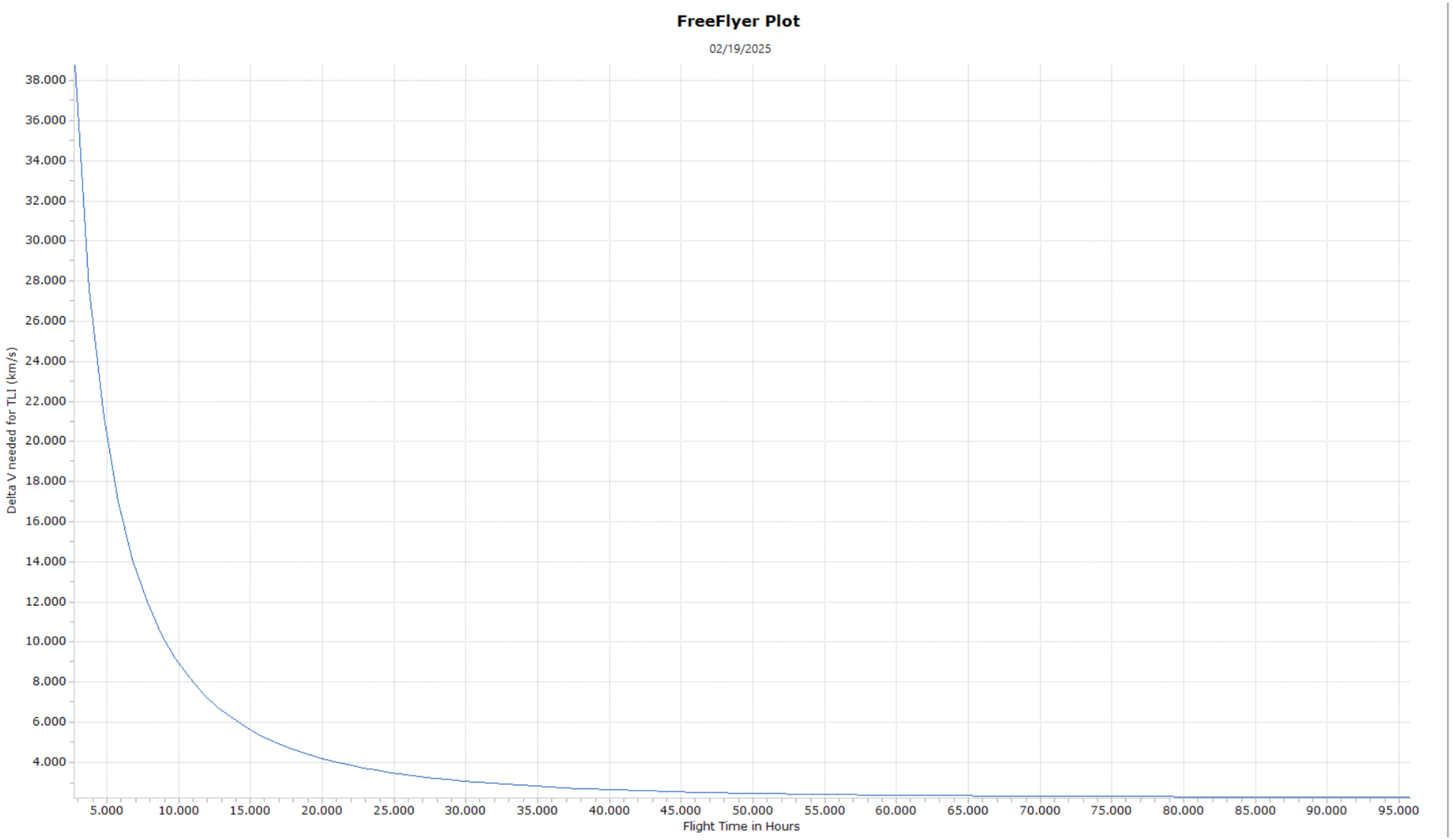
Note: $Departure\ Energy = V_\infty^2 (Hyperbolic\ Escape\ Velocity\ )$

Practical Implementation for the Earth-Moon Transfer

Reduced form of the Porkchop plot as the final date remains fixed – October 28

We see a clear non- linear relationship between the transfer time and required ΔV

In my case I used a transfer time of 96hrs (4days) – Requiring a $\Delta V = 2.257 \frac{km}{s}$



Methods for Entering the Final Orbit

Step 1: Wait until we enter the SOI of the moon

Using the equation: $r_{SOI} = SMA_{Moon} * \left(\frac{m_{Moon}}{M_{Earth}}\right)^{\frac{2}{5}} = 66190.4585\ km$

Change to Moon Reference frame

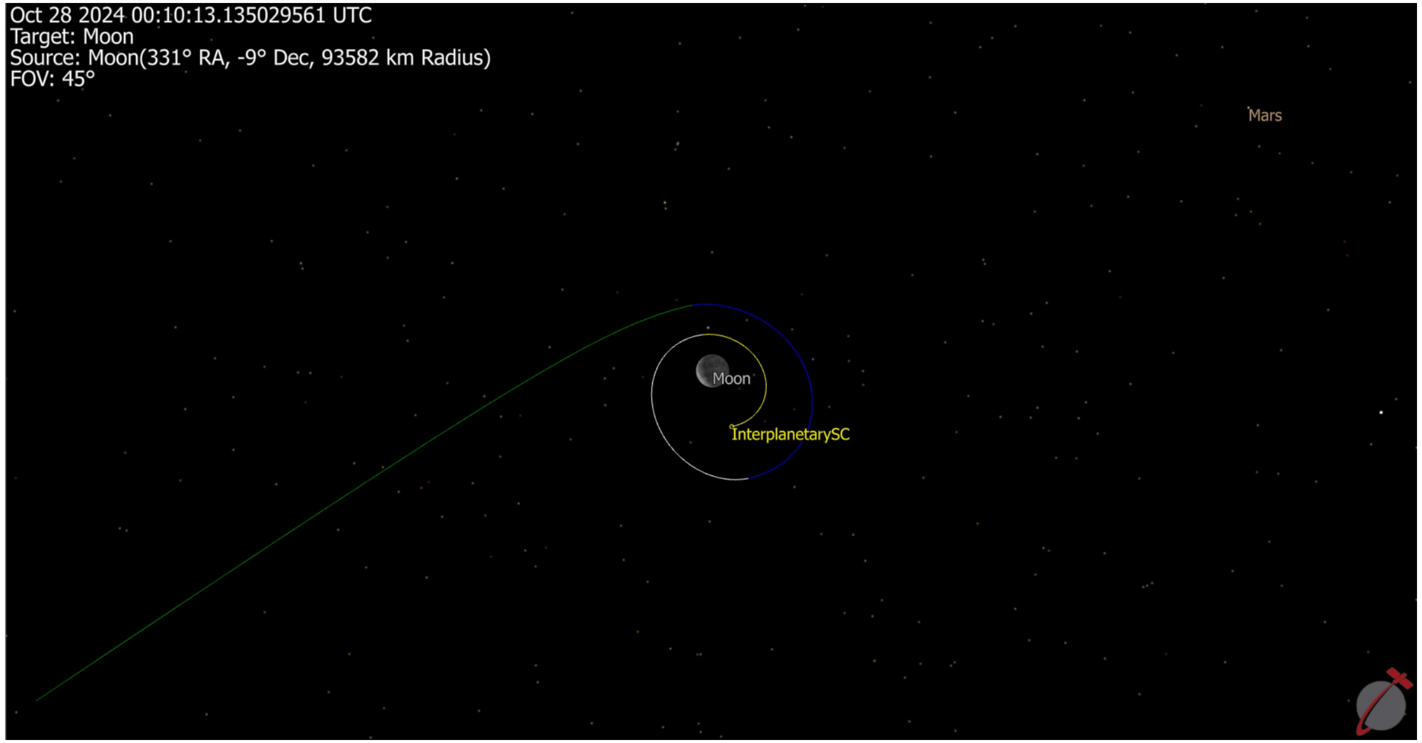
Perform an inclination change to $i = 110°$

Requiring a $\Delta V = 0.499 \frac{km}{s}$ Burn #2

Step 2: Continue to Periapsis of Hyperbolic Orbit

Step 3: Perform an insertion burn to enter with an eccentricity of 0.2 from 7.900

Requiring a $\Delta V = 0.781 \frac{km}{s}$ Burn #3

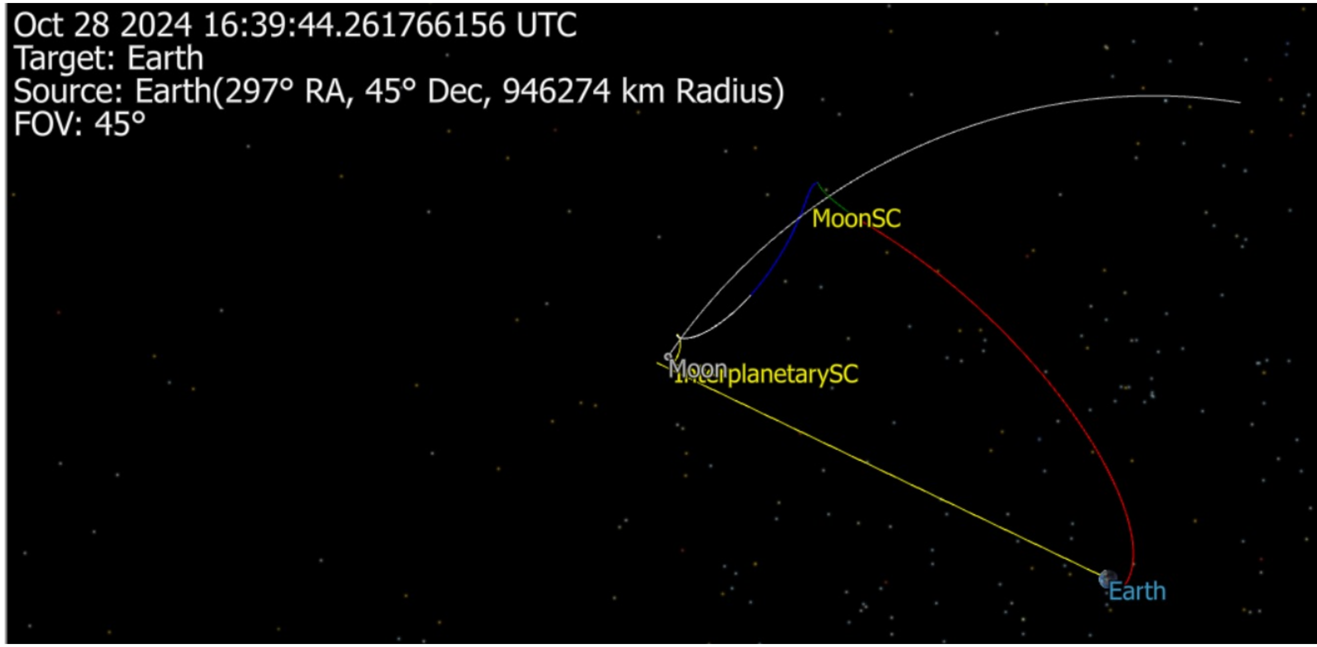


Methods for Entering the Final Orbit

<u>Step 4:</u> Step to the new Orbit Apoapsis
Perform a burn to Raise the periapsis to 4400km
Requiring a $\Delta V = 0.171 \frac{km}{s}$ Burn #2
<u>Step 5:</u> Step to the new Orbit Periapsis
Perform a burn to Raise the Apoapsis to 6600km
Requiring a $\Delta V = 0.259 \frac{km}{s}$ Burn #2

$$Total\ \Delta V = \sum Burns = 3.969 \frac{km}{s}$$

Final Parameters:
Arrival date:
Oct 28 2024 16:39:44.261766156
A: 5510.51 km
E: 0.197823823
I: 110.19 deg
RAAN: 255.80 deg
W: 64.19 deg
TA: 180.00 deg



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